Stability of the two- and three-dimensional kink solutions to the Cahn-Hilliard equation

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(Received 17 December 1996)

We give an analysis of the Cahn-Hilliard equation, which admits both cylindrically and spherically symmetric, stationary kink solutions. Since analytic expressions for these solutions are unobtainable in closed form, we devise an approximate method of solution taking the radius as large and scaling variables in its reciprocal. To lowest order, the solution is that of the one-dimensional kink solution which has been analyzed in earlier work. In this paper we begin by investigating the stability of the cylindrically symmetric kink solution to small perturbations involving angular and z dependence. It is found that the solution is stable to perturbations involving angular variation, but is unstable to a general perturbation. We go on to show that the spherically symmetric kink solution is stable to all small perturbations. [S1063-651X(97)15605-7]

PACS number(s): 64.60.−i, 02.90.+p, 02.30.Mv

I. INTRODUCTION

Pattern formation resulting from a phase transition is observed in alloys, glasses, polymer solutions, and binary liquid mixtures. We are interested in such materials, and consider a two-component system (comprising of components A and B), where a phase transition is induced by quenching the system to below some critical temperature T_c. To study the dynamics of the subsequent concentration of each component, we use the nonlinear equation first proposed by Cahn and Hilliard [1]. Early linear treatments of this equation gave unphysical results, and more involved formulations were preferred to the full nonlinear version. We use this original nonlinear equation in an attempt to ascertain how accurate this continuum model is in describing the stability of particular patterns which have been observed experimentally and numerically. We are encouraged by the results of several authors, including those of Ref. [2], who found that the Cahn-Hilliard equation gives a qualitatively correct description of both the early and late stages of spinodal decomposition. The equation studied is

$$u_t = \nabla^2\left[\frac{dF}{du} - \nabla^2 u\right], \tag{1}$$

where u is the relative concentration of each component, ranging from -1 (all A) to $+1$ (all B). The subscript denotes partial differentiation with respect to t, while F is the free energy which we assume to be given by

$$F(u) = \tfrac{1}{4}(1-u^2)^2. \tag{2}$$

For further physical background, derivation, and discussion of this equation, see [3–5], and references therein. The one-dimensional case is reviewed in [3], [6–8]. A stationary kink solution is found, and in [6,7], it is shown to be stable to perpendicular perturbations of all wavelengths. In [7] it is shown that a more general free energy, leads only to quantitative differences in results.

II. PROBLEM IN CYLINDRICAL GEOMETRY

A. Stationary solution

If we look for a stationary solution to Eq. (1), the equation to be solved is

$$u_e^3 - u_e - \nabla^2 u_e + C = 0, \tag{3}$$

which in cylindrical coordinates, with no θ or z dependence, can be written as

$$\frac{d^2u_e}{dr^2} + \frac{1}{r}\frac{du_e}{dr} - u_e^3 + u_e = C, \tag{4}$$

where C is an arbitrary constant of integration. Unfortunately there is no solution to Eq. (4) in a closed form. We obtain an approximate solution by moving into a new frame of reference, letting $r = R + x$, where R is some large constant, which we take as the value of r where $u_e = 0$. Using this in Eq. (4), we may write

$$\frac{d^2u_e}{dx^2} + u_e - u_e^3 + \varepsilon(1 - \varepsilon x + \varepsilon^2 x^2 + \cdots)\frac{du_e}{dx} = C, \tag{5}$$

where $\varepsilon = 1/R$ and is some small constant. We now expand u_e and C in ε, so that

$$u_e = u_{e0} + \varepsilon u_{e1} + \varepsilon^2 u_{e2} + O(\varepsilon^3),$$
$$C = C_0 + \varepsilon C_1 + \varepsilon^2 C_2 + O(\varepsilon^3). \tag{6}$$

To lowest order in ε, Eq. (5) becomes

$$\frac{d^2u_{e0}}{dx^2} + u_{e0} - u_{e0}^3 = C_0, \tag{7}$$

and for a kink-type solution to exist we *must* set $C_0 = 0$. Then from Eq. (7) it is found that $u_{e0} = \tanh(x/\sqrt{2})$ if we insist that $u_{e0} = 0$ when $x = 0$ $(r = R)$.

To first order in ε, Eq. (5) becomes

$$L_0 u_{e1} = C_1 - \frac{1}{\sqrt{2}}\mathrm{sech}^2\frac{x}{\sqrt{2}}, \tag{8}$$

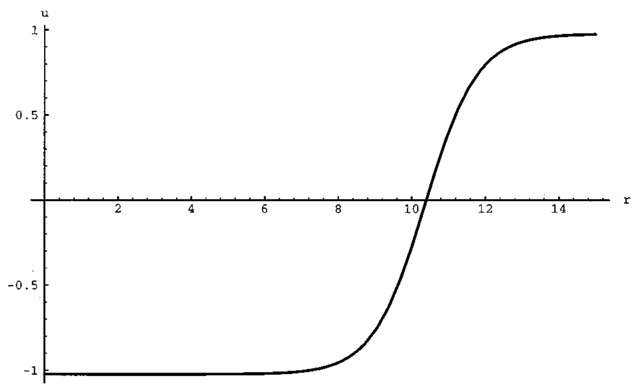

FIG. 1. Approximation to the stationary solution ($R=10.4$).

where $L_0=d^2/dx^2+[3\text{sech}^2(x/\sqrt{2})-2]$. The method used to solve Eq. (8) is that given in [9]. This method is applied using the computer program MATHEMATICA [10]. The solution must be bounded as $r\rightarrow+\infty$ and $r\rightarrow 0$, or as $x\rightarrow\infty$ and $x\rightarrow -R$. Since R is large, we insist that the solution is bounded as $x\rightarrow\pm\infty$, and in doing so find that $C_1=\sqrt{2}/3$, and

$$u_{e1}=-\frac{1}{3\sqrt{2}}\tanh^2\frac{x}{\sqrt{2}}. \tag{9}$$

Thus we are able to write the following approximate expression for the stationary solution:

$$u_e=\tanh\frac{x}{\sqrt{2}}-\frac{1}{3\sqrt{2}R}\tanh^2\frac{x}{\sqrt{2}}+O\left(\frac{1}{R^2}\right). \tag{10}$$

This is plotted in Fig. 1 for $R=10.4$, and, when compared to numerically produced stationary solutions, we find our solution has an error of less than 0.8% (see Fig. 2).

B. Perturbing the stationary solution

To perturb about the stationary solution, we substitute into Eq. (1), $u=u_e+\delta u(r)e^{im\theta+ik_zz+\gamma t}$. Neglecting products of δu, we find that

$$\nabla^2[\nabla^2+(1-3u_e^2)]\delta u=-\gamma\delta u, \tag{11}$$

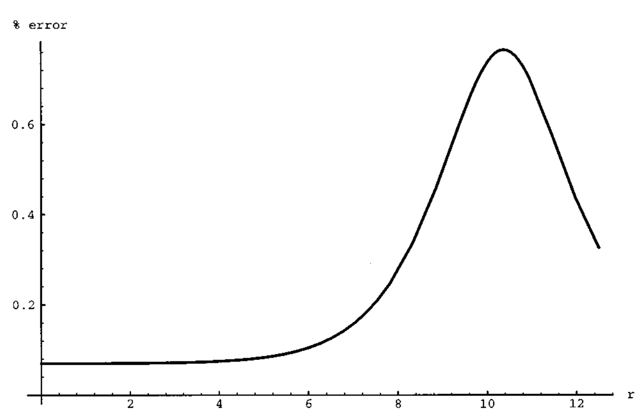

FIG. 2. Percentage error in approximate stationary solution ($R=10.4$).

which is the linear Cahn-Hilliard equation. This has a marginally stable ($\gamma=0$) solution when $m=1$ and $k_z=0$. This is shown by differentiating Eq. (4) with respect to r, and then comparing to Eq. (11) with $\gamma=0$, and $\delta u=du_e/dr$.

Since we have derived an approximate stationary solution by letting $r=x+R$ and taking R as some large constant, we apply the same method to Eq. (11). Begin by considering ∇^2. Now $\nabla^2=d^2/dr^2+(1/r)(d/dr)-m^2/r^2-k_z^2$, but if we make the substitution $r=R+x$, we obtain

$$\begin{aligned}\nabla^2&=\nabla_0^2+\varepsilon\nabla_1^2+\varepsilon^2\nabla_2^2+O(\varepsilon^3)\\&=\left(\frac{d^2}{dx^2}-k^2\right)+\varepsilon\left[\frac{d}{dx}+2x\left(\frac{m}{R}\right)^2\right]\\&\quad-\varepsilon^2x\left[\frac{d}{dx}+3x\left(\frac{m}{R}\right)^2+O(\varepsilon^3)\right],\end{aligned} \tag{12}$$

where $1/R=\varepsilon$ and $k^2=(m/R)^2+k_z^2$. We now go on to order γ and δu in ε,

$$\gamma=\gamma^{(0)}+\varepsilon\gamma^{(1)}+O(\varepsilon^2),\quad \delta u=\delta u^{(0)}+\varepsilon\,\delta u^{(1)}+O(\varepsilon^2). \tag{13}$$

We are now equipped to study Eq. (11) at various orders in ε.

C. Small-*k* analysis

Since R is a large constant, we start by considering m/R as small, and if we also consider k_z as small, then k can be considered small. So for each $\gamma^{(i)}$ and $\delta u^{(i)}$ we introduce an ordering in k, namely,

$$\begin{aligned}\gamma^{(0)}&=\gamma_0^{(0)}+k\gamma_1^{(0)}+O(k^2),\\ \delta u^{(0)}&=\delta u_0^{(0)}+k\,\delta u_1^{(0)}+O(k^2).\end{aligned} \tag{14}$$

It is found that to zeroth order in ε, Eq. (11) is

$$\nabla_0^2L\,\delta u^{(0)}=-\gamma^{(0)}\delta u^{(0)}, \tag{15}$$

where $L=\nabla_0^2+[3\text{sech}^2(x/\sqrt{2})-2]$. Now we have already shown that $\gamma=0$ when $m=1$ and $k_z=0$. This implies that $\gamma^{(0)}=0$ when $k=1/R=\varepsilon$. This effectively means that $k=0$, as k is now included at a higher order in ε. Thus Eq. (15) has a marginally stable solution when $k=0$. This equation has been solved for $\gamma^{(0)}$ for small and large k in [6] and [7], and a Padé approximation found as a complete expression in [7]. For small k,

$$\gamma^{(0)}=-\frac{\sqrt{2}}{3}k^3-\tfrac{11}{18}k^4+O(k^5), \tag{16}$$

and it is also found that $\delta u_0^{(0)}=\text{sech}^2(x/\sqrt{2})$, $\delta u_1^{(0)}=0$, and $\delta u_2^{(0)}=\tfrac{1}{3}$.

We now use Eq. (15) to find the adjoint operator ψ. Begin by multiplying Eq. (15) by ψ and integrating over all space,

$$\int_{-\infty}^{\infty}\psi\nabla_0^2L\,\delta u^{(0)}dx=-\gamma^{(0)}\int_{-\infty}^{\infty}\psi\delta u^{(0)}dx. \tag{17}$$

The left-hand side of Eq. (17) is now integrated by parts to leave

$$\int_{-\infty}^{\infty}\delta u^{(0)}L\nabla_0^2\psi dx=-\gamma^{(0)}\int_{-\infty}^{\infty}\delta u^{(0)}\psi dx, \tag{18}$$

and thus the adjoint equation is

$$L\nabla_0^2\psi=-\gamma^{(0)}\psi. \tag{19}$$

If we operate on this adjoint equation with ∇_0^2, comparison to Eq. (15) shows that

$$\nabla_0^2\psi=\delta u^{(0)}. \tag{20}$$

To first order in ε, the linear equation (11) is

$$\nabla_0^2L\,\delta u^{(1)}+\gamma^{(0)}\delta u^{(1)}=\nabla_0^2(6u_{e0}u_{e1}-\nabla_1^2)\delta u^{(0)}-\nabla_1^2L\,\delta u^{(0)}-\gamma^{(1)}\delta u^{(0)}. \tag{21}$$

Using a similar technique to that employed in [6] for determining the stability of the one-dimensional kink solution, we multiply Eq. (21) by ψ and integrate over all space. With some manipulation, it can be shown that

$$\gamma^{(1)}=[-6\gamma^{(0)}\langle u_{e0}u_{e1}(\delta u^{(0)})^2\rangle+\gamma^{(0)}\langle\delta u^{(0)}\nabla_1^2\delta u^{(0)}\rangle-\langle L\,\delta u^{(0)}\nabla_1^2L\,\delta u^{(0)}\rangle]/\langle\delta u^{(0)}L\,\delta u^{(0)}\rangle, \tag{22}$$

where the angled brackets $\langle\rangle$, denote integration over all x. We now observe that in the numerator of Eq. (22) all of the integrals are of odd functions, and so $\gamma^{(1)}=0$ for all k. This result is used in Eq. (21), and it is found that $\delta u_0^{(1)}=-(\sqrt{2}/3)\mathrm{sech}^2(x/\sqrt{2})\tanh(x/\sqrt{2})$, $\delta u_1^{(1)}=0$, and $L_0\delta u_2^{(1)}=-2x\,\mathrm{sech}^2(x/\sqrt{2})-(\sqrt{2}/3)\tanh(x/\sqrt{2})$.

Now Eq. (11) to second order in ε is multiplied by ψ, and integrated over all space, to give

$$\frac{\langle\delta u^{(0)}L\,\delta u^{(0)}\rangle}{\gamma^{(0)}}\gamma^{(2)}=\langle\psi\nabla_1^2L\,\delta u^{(1)}\rangle-\langle\psi\nabla_1^2(6u_{e0}u_{e1}-\nabla_1^2)\delta u^{(0)}\rangle-\langle\delta u^{(0)}(6u_{e0}u_{e1}-\nabla_1^2)\delta u^{(1)}\rangle+\langle\psi\nabla_2^2L\,\delta u^{(0)}\rangle-\langle\delta u^{(0)}\times(6u_{e0}u_{e2}+3u_{e1}^2-\nabla_2^2)\delta u^{(0)}\rangle. \tag{23}$$

Note here that all the integrals are of even functions, and so will contribute to the answer.

We now consider Eq. (23) to the lowest two orders in k. To do this we need to know the adjoint function ψ. We can write Eq. (20) as

$$\psi=(\nabla_0^2)^{-1}\delta u^{(0)}, \tag{24}$$

and so require the Green's function for the operator $(\nabla_0^2)^{-1}$. It is found to be given by

$$G(x;x')=-\frac{e^{-k|x-x'|}}{2k}, \tag{25}$$

and so we write

$$\psi=-\frac{1}{2k}\int_{-\infty}^{\infty}\delta u^{(0)}(x')e^{-k|x-x'|}dx'=-\frac{\sqrt{2}}{k}+f(x)+O(k), \tag{26}$$

where $f(x)$ is some function of x.

We now consider each definite integral of Eq. (23) separately to the two lowest orders in k. The first such integral is

$$\begin{aligned}\frac{\langle\delta u^{(0)}L\,\delta u^{(0)}\rangle}{\gamma^{(0)}}&=\frac{1}{\gamma^{(0)}}\langle(\delta u_0^{(0)}+k^2\delta u_2^{(0)})(L_0-k^2)\times(\delta u_0^{(0)}+k^2\delta u_2^{(0)}+k^3\delta u_3^{(0)})\rangle\\&=\frac{1}{\gamma^{(0)}}\langle\delta u_0^{(0)}(k^2L_0\delta u_2^{(0)}+k^3L_0\delta u_3^{(0)}-k^2\delta u_0^{(0)})\rangle\\&=-\frac{k^2}{\gamma^{(0)}}\langle(\delta u_0^{(0)})^2\rangle=-\frac{4\sqrt{2}k^2}{3\gamma^{(0)}}.\end{aligned} \tag{27}$$

Now it can be shown that

$$\langle\psi\nabla_1^2L\,\delta u^{(1)}\rangle=v_1+k(v_2-\tfrac{4}{3})\times\langle\psi\nabla_1^2(6u_{e0}u_{e1}-\nabla_1^2)\delta u^{(0)}\rangle=v_1+k(v_2-\tfrac{4}{3}), \tag{28}$$

where v_1 and v_2 are some constants. This result becomes obvious if we consider Eq. (21) to the lowest two orders in k, namely,

$$\nabla_0^2L\,\delta u^{(1)}=\nabla_0^2(6u_{e0}u_{e1}-\nabla_1^2)\delta u^{(0)}, \tag{29}$$

and, neglecting constants of integration, this becomes

$$L\,\delta u^{(1)}=(6u_{e0}u_{e1}-\nabla_1^2)\delta u^{(0)}, \tag{30}$$

which validates the result in Eq. (28). We proceed to consider the next definite integral, namely,

$$\langle\delta u^{(0)}(6u_{e0}u_{e1}-\nabla_1^2)\delta u^{(1)}\rangle=\langle\delta u_0^{(0)}(6u_{e0}u_{e1}-\nabla_1^2)\delta u_0^{(1)}\rangle=\frac{16\sqrt{2}}{63}+O(k^2), \tag{31}$$

where there is no contribution at order k because $\delta u_1^{(0)}=\delta u_1^{(1)}=0$. The next definite integral to consider is

$$\langle\psi\nabla_2^2L\,\delta u^{(0)}\rangle=\left\langle\left(-\frac{\sqrt{2}}{k}+f\right)\left[-x\frac{d}{dx}-3x^2\left(\frac{m}{R}\right)^2\right]\times(L_0-k^2)(\delta u_0^{(0)}+k^2\delta u_2^{(0)})\right\rangle=\sqrt{2}k\left\langle x\frac{d}{dx}\left(-\frac{2}{3}\right)\right\rangle=O(k^2). \tag{32}$$

Finally,

$$\langle \delta u^{(0)}(6u_{e0}u_{e2}+3u_{e1}^2-\nabla_2^2)\delta u^{(0)}\rangle$$
$$=\langle \delta u_0^{(0)}(6u_{e0}u_{e2}+3u_{e1}^2-\nabla_2^2)\delta u_0^{(0)}\rangle$$
$$=-\frac{100\sqrt{2}}{63}+O(k^2), \tag{33}$$

where again there is no contribution at order k because $\delta u_1^{(0)}$ is equal to zero.

We put these results into Eq. (23) to find that to the first two orders in k,

$$\gamma^{(2)}=-\frac{\gamma^{(0)}}{k^2}=\frac{\sqrt{2}}{3}k+\frac{11}{18}k^2. \tag{34}$$

We continue to consider the linear Cahn-Hilliard equation (11) to higher orders in ε. To order ε^3, we again multiply the equation by ψ and integrate over all x. Many of the definite integrals are zero due to the functions being odd, and we are left with $\gamma^{(3)}=0$ for all k.

Using the same procedure on the linear equation to fourth order in ε, we find that

$$\gamma^{(4)}=-\frac{k}{4}\left(\langle \psi g(x)\rangle+\frac{2}{3}\langle \delta u_0^{(2)}\rangle\right), \tag{35}$$

where $g(x)$ is some known function. Thus from this it is clear that $\gamma_0^{(4)}=0$.

We now combine our growth rate results and find that

$$\gamma=-\frac{\sqrt{2}}{3}k^3-\frac{11}{18}k^4+O(k^5)+\varepsilon^2\left(\frac{\sqrt{2}}{3}k+\frac{11}{18}k^2+O(k^3)\right)$$
$$+\varepsilon^4 O(k)+O(\varepsilon^5)$$
$$=-\frac{\sqrt{2}}{3}\left(k_z^2+\frac{m^2}{R^2}\right)^{1/2}\left(k_z^2+\frac{m^2-1}{R^2}\right)$$
$$\times\left(1+\frac{11}{6\sqrt{2}}\left(k_z^2+\frac{m^2}{R^2}\right)^{1/2}\right)+O\left(k^5,\frac{k^3}{R^2},\frac{k}{R^4},\frac{1}{R^5}\right). \tag{36}$$

Thus clearly a marginally stable state exists when $m=1$ and $k_z=0$, and when $m=0$ and $k_z=0$, in agreement with general statements made above. Note that it is unphysical to consider perturbations where $m=0$ and $k_z=0$, since here the perturbations are purely radial, and so break the law relating to conservation of mass in the system. For $m>1$, $\gamma<0$, implying stability.

However, for $m\equiv 0$, we see that γ can be positive and hence the equilibrium is unstable if $k_z^2R^2<1$. This result is confirmed if we return to the analysis of Eq. (11), insist that $m\equiv 0$, and introduce an ordering such that $k\equiv k_z=\varepsilon\overline{k_z}$. If this is performed, Eq. (12) becomes

$$\nabla^2=\frac{d^2}{dx^2}+\varepsilon\frac{d}{dx}-\varepsilon^2\left(x\frac{d}{dx}+\overline{k_z^2}\right)+\varepsilon^3x^2\frac{d}{dx}+O(\varepsilon^4). \tag{37}$$

So, to lowest order in ε, Eq. (11) becomes

$$\frac{d^2}{dx^2}L_0\delta u^{(0)}=-\gamma^{(0)}\delta u^{(0)}, \tag{38}$$

which has the solution $\delta u^{(0)}=\mathrm{sech}^2(x/\sqrt{2})$, and $\gamma^{(0)}=0$. From Eq. (20) it is seen that the adjoint function ψ is no longer bounded. This implies that we can no longer multiply the linear equation, at each order in ε, by ψ, and integrate over all x, as we have done previously. Instead we use an asymptotic matching method similar to that used in [7]. Here we consider the asymptotic solution of the linear equation (11), at each order in ε. It is found that $\gamma^{(0)}=\gamma^{(1)}=\gamma^{(2)}=0$, and neglecting terms of order $e^{-\sqrt{2}x}$ and higher, that

$$\lim_{x\to\infty}\delta u=\overline{\delta u}\propto 1-\frac{3\gamma^{(3)}\varepsilon x}{\sqrt{2}(1-\overline{k_z^2})}+O(\varepsilon^2). \tag{39}$$

We now note that

$$\lim_{x\to\infty}\nabla^2=\frac{d^2}{dx^2}-\varepsilon^2\overline{k_z^2}. \tag{40}$$

If we also assume that $\lim_{x\to\infty}\delta u=\overline{\delta u}\propto e^{\lambda x}$, we find that asymptotically Eq. (11) becomes

$$(\lambda^2-\varepsilon^2\overline{k_z^2})(\lambda^2-2+O(\varepsilon))=0. \tag{41}$$

In this method we only consider terms which decay slowly as $x\to\infty$, and so we choose $\lambda=-\varepsilon\overline{k_z}$. Thus we can write

$$\overline{\delta u}\propto e^{-\varepsilon\overline{k_z}x}=1-\varepsilon\overline{k_z}x+O(\varepsilon^2). \tag{42}$$

Comparison of this result to that in Eq. (39) shows that

$$\gamma=\frac{\sqrt{2}\overline{k_z}}{3R^3}(1-\overline{k_z^2})+O\left(\frac{1}{R^4}\right), \tag{43}$$

which is in agreement with Eq. (36). From Eq. (43) it can be shown that to lowest order in $1/R$, the maximum growth rate is achieved when $k_z=1/(\sqrt{3}R)$.

Also if $k_z=0$, and m/R is written as εm rather than k, it is found that, up to order ε^2, ∇^2 is as given in Eq. (37) but with $\overline{k_z}$ replaced by m. Thus again we apply the asymptotic method used in [7], and find that to lowest order in $1/R$ the growth rate of perturbations is given by Eq. (43), with $\overline{k_z}$ replaced by m. This is verified by setting $k_z=0$ in Eq. (36). Note here, that γ is never positive, since m only takes integer values. The above analysis confirms the general form for γ given by Eq. (36).

So we have shown that the stationary solution is stable to perturbations which involve any angular variation ($m\neq 0$, $k_z\neq 0$), but, for a radial perturbation ($m=0$) varying along the axis of the cylinder, the stationary solution is unstable if $k_zR<1$. The growth rate γ, for a particular value of R, is plotted in Fig. 3.

D. Large-*k* analysis

In this section we consider solutions of Eq. (11) for k large. Unfortunately the analysis is not trivial, and so we must break the problem into k_z large and (a) $m=0$, (b) m of order 1, (c) m large.

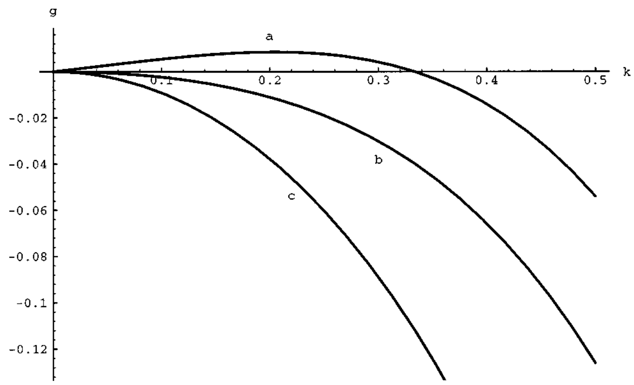

FIG. 3. Growth rate $[g=\gamma(m,k_z)-\gamma(m,0)]$ against $k=k_z$ for $R=3$, and three different values of m. Graph a corresponds to $m=0$. Graph b corresponds to $m=1$; graph c corresponds to $m=2$.

For $m=0$, we find that the problem reverts back to the one-dimensional case, with a correction at order ε^2. It is found that

$$\frac{\gamma}{k_z^4}=-1+\frac{1}{k_z^2}\left(\gamma_c^{(0)}+\frac{\gamma_c^{(2)}}{R^2}\right)+O\left(\frac{1}{k_z^3},\frac{1}{R^3}\right), \tag{44}$$

where $\gamma_c^{(0)}=(3-\sqrt{13})/2$, and in principle $\gamma_c^{(2)}$ can be found.

For m of order 1, we find that Eq. (44) holds, but with a slight correction at order ε^2 due to the finite value of m. It is found that

$$\frac{\gamma}{k_z^4}=-1+\frac{1}{k_z^2}\left(\gamma_c^{(0)}+\frac{\gamma_c^{(2)}+2m^2}{R^2}\right)+O\left(\frac{1}{k_z^3},\frac{1}{R^3}\right), \tag{45}$$

where $\gamma_c^{(0)}$ is as above.

For m large we consider m/R to be of order 1. This alters the zeroth-order result in ε, which becomes

$$\frac{\gamma^{(0)}}{k_z^4}=-1+\frac{\gamma_c^{(0)}}{k_z^2}, \tag{46}$$

where $\gamma_c^{(0)}=(3-\sqrt{13})/2-2(m/R)^2$. From this we see that for m large the stability is increased. Thus we see from these three cases that the stationary solution is stable for k_z large along with any value of m.

III. PROBLEM IN SPHERICAL GEOMETRY

We begin by looking for a stationary solution to Eq. (1). The equation to be solved is

$$\frac{d^2u_e}{dr^2}+\frac{2}{r}\frac{du_e}{dr}-u_e^3+u_e=C. \tag{47}$$

Comparison with Eq. (4) shows how similar this is to the cylindrically symmetric problem. Again we make the substitution $r=R+x$, and find that there is no difference to zeroth order in ε. It can be shown that

$$u_e=\tanh\frac{x}{\sqrt{2}}-\frac{\sqrt{2}}{3R}\tanh^2\frac{x}{\sqrt{2}}+O\left(\frac{1}{R^2}\right). \tag{48}$$

To perturb about this stationary solution in spherical geometry, we make the substitution

$$u=u_e+\delta u(r)(\sin^{|m|}\theta)\mathrm{F}_l(\cos\theta)e^{\gamma t+im\phi} \tag{49}$$

into Eq. (1). Here F_l is the associated Legendre function, where l takes the values $|m|,|m+1|,|m+2|,\ldots$. The subsequent equation is the linear Cahn-Hilliard equation, namely

$$\nabla^2[\nabla^2+(1-3u_e^2)]\delta u=-\gamma\delta u, \tag{50}$$

which is as in Eq. (11), but now

$$\nabla^2=\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{\alpha}{r^2}, \tag{51}$$

where $\alpha=l(l+1)$ and u_e is now the *spherically* symmetric stationary solution. It can be shown that Eq. (50) has a marginally stable solution when $\alpha=2$ $(l=1)$. Clearly the problem is similar to that in cylindrical geometry, and, using either the consistency condition or asymptotic method, it is found that the growth rate is given by

$$\gamma=-\frac{\sqrt{2}}{3R^3}\sqrt{l(l+1)}(l-1)(l+2)+O\left(\frac{1}{R^4}\right), \tag{52}$$

which is valid for $\sqrt{l(l+1)}<R$. We see that, if $l=1$, $\gamma=0$, which verifies the statement above. Also if $l=0$, $\gamma=0$, which, as shown for cylindrical geometry, corresponds to the unphysical case of a purely radial perturbation. For $\sqrt{l(l+1)}\gg R$, it can be shown that, to lowest order,

$$\gamma=-\frac{l^2(l+1)^2}{R^4}. \tag{53}$$

From this we see that $\gamma<0$ for $\sqrt{l(l+1)}\simeq R$, and so, for $l>1$, the spherically symmetric solution is stable.

IV. CONCLUSIONS

We have considered the existence and stability of both cylindrically and spherically symmetric kink solutions to the Cahn-Hilliard equation. In both cases, this is done by considering the radius R of the solution as large, and scaling the variables in its reciprocal. To lowest order we effectively assume that $R\to+\infty$, and so the curvature of the solution tends to zero. Thus here we revert back to the one-dimensional case, which has a stationary kink solution. This has been shown, in earlier papers, to be stable. We go on to higher orders in $1/R$, and γ is determined via two independent methods, namely, a consistency condition and an asymptotic method, similar to those applied to the one-dimensional equation in [6] and [7], respectively.

We show that the cylindrically symmetric solution is stable for all m and k_z, of angular and z-dependent perturbation, unless $m=0$ and $k_z^2R^2<1$. Thus for a *general* perturbation, this stationary solution is *unstable*. In contrast, it is found that the spherical equivalent is always stable. We assume that the unstable cylindrically symmetric solution decays to one or more spherically symmetric states. A similar behavior has been found for a different physical situation by Frycz, Infeld, and Samson [11]. In summary, our analysis suggests that under certain conditions, the cylindrically symmetric solution will pass through an unstable state, and tend to one or more stable, spherically symmetric solutions.

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